Stochastic dynamics in systems with unidirectional delay coupling: Two-state description

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We study stochastic dynamics of two-state particles coupled unidirectionally with delay. We give exact results for the stationary distribution function p_{st} and the time correlation function (TCF) when the system consists of two ($N=2$) and three ($N=3$) particles. Based on these results, effects of delay are discussed and compared with the $N=1$ case, studied by Tsimring and Pikovsky [Phys. Rev. Lett. **87**, 250602 (2001)]. Next, we consider the general N-particle system, for which we give exact expressions for p_{st} and the TCF, which are inferred based on the $N=2$ and $N=3$ solutions and then confirmed via detailed arguments. It is pointed out that the stationary state is mapped to Ising spin model with ferro(antiferro)magnetic interaction when delay feedback is positive (negative).

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I. INTRODUCTION

Much attention is focused on a system with noise and delay. For linear systems, which is described by a linear Langevin equation with delay feedback, we observed considerable progresses in our understanding of the properties of the system and some exact results are obtained for the (stationary) distribution function, the time correlation function (TCF) [1,2], and the energy dissipation rate [3]. Exact results are useful because they often serve as a starting point for perturbational treatments [4].

For nonlinear stochastic systems with delay, we have interesting studies for biological models, such as pupils dynamics [5], cell differentiation [6], and human motor control [7] if we mention a few. As a typical nonlinear system, we consider in this paper the delay stochastic bistable system, which is gathering a lot of attention partly from the general interest in the interplay among nonlinearity, noise, and delay and partly in connection with a model for vertical-cavity surface-emitting lasers (VCSELs) with optoelectronic feedback [8].

It is notable that some experimental results such as the power spectrum and the mean residence time for the VCSEL could be theoretically explained [9] based on the two-state description of the delay stochastic bistable model, first proposed by Tsimring and Pikovsky (TP) [10]. That is, TP reduced the original continuum description based on the nonlinear Langevin equation with delay to a master equation, which is touched on in Sec. II to make this paper self-contained. The master equation could be solved to give exact results on the TCF and its power spectrum [10], thus revealing effects of delay such as the coherence resonance and also the stochastic resonance (SR) in the linear response to the external periodic force.

The two-state model had already played an important role in clarifying physics behind the nonlinear Langevin model in relation to SR [11]. An analytic result was given [12] for the signal to noise ratio and this greatly deepened our understanding of the problem. It is noted also that simulations, if the system is a coupled one with many degrees of freedom, are greatly facilitated by the introduction of the two-state description.

The purpose of this paper is to extend the two-state approach to the unidirectionally coupled double-well systems and to study properties of the systems, such as the stationary many-body distribution function and the TCF, both analytically and numerically. In Sec. II we present the master equation for the system, consisting of unidirectionally coupled two-state N particles with delay. In Sec. III the master equation are solved for the case $N=2$ and $N=3$ to give exact results for the stationary distribution and the TCF, after the case $N=1$ [10] is briefly reviewed. It is noted that the stationary distribution function depends on the delay time τ, in sharp contrast with the case $N=1$. We also plot explicitly the stationary distribution and the TCF and discuss properties of the unidirectionally coupled two-state systems for $N=2$ and $N=3$. In Sec. IV we give analytic solutions to the general N-body problems, inferred from the exact solutions for cases $N=2$ and $N=3$. This is justified by detailed arguments. Finally this paper is concluded with some remarks.

II. BISTABLE SYSTEMS WITH DELAY: TWO-STATE DESCRIPTION

As a typical bistable system with delay let us consider the Langevin equation

$$dx(t)=[x(t)-x^3(t)+\epsilon x(t-\tau)]dt+\sqrt{2D}dw(t), \qquad (1)$$

where τ is the delay and $dw(t)$ denotes the Gaussian noise with the relation

$$\langle dw(t)dw(t')\rangle=\delta_{t,t'}dt. \qquad (2)$$

Here the Kronecker delta function $\delta_{t,t'}$ is zero if the time segments dt and dt' are different and one if the two time segments are the same. The delay force $\epsilon x(t-\tau)$ on the right-hand side of Eq. (1) represents an attractive(repulsive) force from $x(t-\tau)$ if $\epsilon>0(<0)$.

It is well known that analytic approaches to Eq. (1) are difficult due to the intrinsic nonlinearity of Eq. (1). Also from a simulational point of view, it is very time consuming, especially when we consider the N-particle extension of Eq. (1) to be introduced in Sec. II, to solve the stochastic differential Eq. (1) [or especially Eq. (8) below] for long time to obtain a time correlation function

$$\psi(t) = \langle x(0)x(t)\rangle. \tag{3}$$

Under this circumstance, the two-state description of Eq. (1) achieved considerable success, enabling one to have an analytic result for $\psi(t)$ defined by

$$\psi(t) = \langle \sigma(0)\sigma(t)\rangle, \tag{4}$$

with $\sigma(t)$ taking either the value 1 or -1 depending on $x(t)$. Let us first turn to this two-state description for the case $N=1$ [10].

A. Two-state description of the Langevin model ($N=1$)

If $p(1,t)=1-p(-1,t)$ denotes the probability that $\sigma(t)=1$, we have a master equation

$$dp(1,t)/dt = -T(1\to -1,t)p(1,t) + T(-1\to 1,t)p(-1,t), \tag{5}$$

where $T(1\to -1,t)$ and $T(-1\to 1,t)$ denote the transition probabilities from the state 1 to -1 and from -1 to 1 at time t, respectively, and these are expressed as [10]

$$T(1\to -1,t) = \gamma_1 p(1,t-\tau) + \gamma_2 p(-1,t-\tau),$$

$$T(-1\to 1,t) = \gamma_2 p(1,t-\tau) + \gamma_1 p(-1,t-\tau). \tag{6}$$

If there were no delay force, we would have $T(1\to -1,t)=T(-1\to 1,t)\equiv\gamma_0$ since the double-well potential in Eq. (1) is symmetric [13]. Owing to the interaction due to delay, a particle with $\sigma(t)=1$ hops to the state -1 more easily when $\sigma(t-\tau=-1)$ than when $\sigma(t-\tau=1)$, where ϵ is assumed to be positive. Thus γ_2 is larger than γ_1. When the noise intensity D is small and the Kramers expression for the hopping rate is valid, γ_1 and γ_2 can be expressed in terms of the parameters appearing in the Langevin model [Eq. (1)] [see Eq. (3) of Refs. [10,14]]. Since we are mainly interested in the two-state model, we will not pursue this connection any more [14]. From Eqs. (5) and (6) we have [10]

$$dp(1,t)/dt = -(\gamma_1+\gamma_2)p(1,t) + (\gamma_2-\gamma_1)p(1,t-\tau) + \gamma_1 \tag{7}$$

based on which we can discuss the TCF [Eq. (4)].

B. Unidirectional coupling of N particles

We now consider a simple generalization of the model described by the Langevin Eq. (1) and the corresponding master Eq. (7). In Eq. (1) the output signal is fed back to itself. If we consider that the particle i feeds forward its output to the system $(i+1)$. Then under the cyclic arrangement of particles, we find that $x_i(t)$ of the particle i is described by

$$dx_i(t) = [x_i(t) - x_i^3(t) + \epsilon x_{i-1}(t-\tau)]dt + \sqrt{2D}dw_i(t), \tag{8}$$

where $dw_i(t)$ denotes noise with the relation

$$\langle dw_i(t)dw_j(t')\rangle = \delta_{i,j}\delta_{t,t'}dt, \tag{9}$$

with $x_0=x_N$. The coupling in Eq. (8) may be called unidirectional.

Following the similar reasoning as TP [10], we have a master equation for the probability that $\sigma_i=\pm 1$ at time t, $p_i(\pm 1,t),(i=1,2,\ldots,N)$,

$$dp_i(1,t)/dt = -(\gamma_1+\gamma_2)p_i(1,t) + (\gamma_2-\gamma_1)p_{i-1}(1,t-\tau) + \gamma_1, \tag{10}$$

with $p_0(\pm 1,t)=p_N(\pm 1,t)$.

III. p_{st} AND TCF OF THE TWO-STATE SYSTEMS IN THE CASES $N=2$ AND $N=3$

A. Two-state systems in the case $N=1$ [10]

Before proceeding to our problem of unidirectional coupling with delay, we briefly discuss how one can calculate the TCF from Eq. (7), in a way generalizable to the case $N\geq 2$ [Eq. (10)]. First let us express $\psi(t)$ [Eq. (4)] as

$$\psi(t) = \sum_{s_0,s_1} s_1 s_0 p(s_0,0;s_1,t) = \sum_{s_0,s_1} s_1 s_0 p_{st}(s_0)p(s_1,t|s_0), \tag{11}$$

where $p(s_0,0;s_1,t)$ denotes the two-time joint probability for $\sigma(t=0)=s_0$ and $\sigma(t)=s_1$ and $p_{st}(s_0)$ is the probability for $\sigma=s_0$ in a stationary state. Similarly $p(s_1,t|s_0)$ is the conditional probability for $\sigma(t)=s_1$ given that $\sigma(t=0)=s_0$. It is important to stress here that the system is assumed to be in a stationary state, which is realized long after one starts a (Monte Carlo) simulation of the dynamics, corresponding to Eq. (7).

Basically we can use two invariance relations to simplify the calculations of the TCF. One is the invariance due to system symmetry for inversion $s\to -s$. From this we have immediately

$$p_{st}(s) = p_{st}(-s) = 1/2, \tag{12}$$

$$p(1,t|1) = p(-1,t|-1), \quad p(1,t|-1) = p(-1,t|1). \tag{13}$$

These relations simplify Eq. (11) to

$$\psi(t) = 2p(1,t|1) - 1. \tag{14}$$

The other invariance relation comes from the stationarity of the process $p(s_0,0;s_1,t)=p(s_0,-t;s_1,0)$ for arbitrary $s_0=\pm 1$ and $s_1=\pm 1$. This relation is rewritten as

$$p_{st}(s_0)p(s_1,t|s_0) = p_{st}(s_1)p(s_0,-t|s_1). \tag{15}$$

With use of the symmetry relations (12), (13), and (15), we immediately see that the conditional probability $p(s_1,t|s_0)$ is even in t,

$$p(s_1,t|s_0) = p(s_1,-t|s_0). \tag{16}$$

From this it is seen that $\psi(t)$ is an even function of t as it should be. To calculate $\psi(t)$, it is only necessary to note that the conditional probability $p(1,t|1)$ is obtained as the (even) solution to Eq. (7), which satisfies the initial condition $p(1,0)=1$. Analytic solution to Eq. (7) is available and detailed discussions on the TCF and the residence time distribution function are given in [9,10].

B. Two-state system in the case $N=2$: Theory

We can generalize the treatment above to study the TCF and the stationary distribution function for the case $N=2$. We have from Eq. (10) that

$$dp_1(1,t)/dt = -(\gamma_1+\gamma_2)p_1(1,t) + (\gamma_2-\gamma_1)p_2(1,t-\tau) + \gamma_1,$$

$$dp_2(1,t)/dt = -(\gamma_1+\gamma_2)p_2(1,t) + (\gamma_2-\gamma_1)p_1(1,t-\tau) + \gamma_1. \tag{17}$$

In what follows, we try to calculate the self- and cross-correlation functions

$$\psi_s(t) = \langle \sigma_1(t)\sigma_1(0)\rangle = \sum_{s_1^0,s_2^0,s_1} s_1 s_1^0 p_{st}(s_1^0,s_2^0)p(s_1,t|s_1^0,s_2^0),$$

$$\psi_c(t) = \langle \sigma_1(t)\sigma_2(0)\rangle = \sum_{s_1^0,s_2^0,s_1} s_1 s_2^0 p_{st}(s_1^0,s_2^0)p(s_1,t|s_1^0,s_2^0), \tag{18}$$

and the stationary distribution function $p_{st}(\sigma_1=s_1,\sigma_2=s_2) \equiv p_{st}(s_1,s_2)$.

1. Some symmetry relations

We consider important symmetry relations necessary to make the calculations of $\psi_s(t)$ and $\psi_c(t)$ as simple as possible. To simplify notations, we define four states A, B, C, and D by

$$A:(\sigma_1=\sigma_2=1), \quad B:(\sigma_1=1,\sigma_2=-1),$$

$$C:(\sigma_1=-1,\sigma_2=1), \quad D:(\sigma_1=\sigma_2=-1). \tag{19}$$

Since our system is symmetric with respect to the inversion $s_1,s_2 \to -s_1,-s_2$ we have

$$p_{st}(A) = p_{st}(D), \quad p_{st}(B) = p_{st}(C). \tag{20}$$

Similarly, we have

$$a \equiv p(B,t|B) = p(C,t|C), b \equiv p(A,t|D) = p(D,t|A),$$

$$d \equiv p(A,t|A) = p(D,t|D), e \equiv p(B,t|C) = p(C,t|B),$$

$$x \equiv p(B,t|D) = p(C,t|D) = p(B,t|A) = p(C,t|A),$$

$$y \equiv p(A,t|B) = p(A,t|C) = p(D,t|B) = p(D,t|C), \tag{21}$$

where use is made of the symmetry $1,2 \leftrightarrow 2,1$ in deriving, e.g., $p(B,t|D)=P(C,t|D)$. In passing we note from the conservation of probability that

$$d+b+2x=1, \quad a+e+2y=1. \tag{22}$$

Other useful relations can be derived from the stationarity condition

$$p_{st}(s_0^1,s_0^2)p(s_1^1,s_1^2,t|s_0^1,s_0^2) = p_{st}(s_1^1,s_2^1)p(s_0^1,s_0^2,-t|s_1^1,s_1^2). \tag{23}$$

From $p_{st}(A)p(D,t|A)=p_{st}(D)p(A,-t|D)$ and Eq. (20) we have $p(A,t|D)=p(D,-t|A)$, resulting in the fact that b in Eq. (21) is even in t. By similar arguments we can show that conditional probability functions a,d,e in Eq. (21) are also even in t.

2. Stationary distribution functions

We turn to $p_{st}(s_1,s_2)$. After summing over s_0^1,s_0^2 on both sides of Eq. (23), we have the eigenvalue problem

$$\sum_{s_0^1,s_0^2} p_{st}(s_0^1,s_0^2)p(s_1^1,s_1^2,t|s_0^1,s_0^2) = p_{st}(s_1^1,s_2^1). \tag{24}$$

With use of relations (20)–(22) we only have the following equation from Eq. (24)

$$p(B,t|A)p_{st}(A) = p(A,t|B)p_{st}(B). \tag{25}$$

From this, we have

$$p(B,t|A) = p(B,t;A,0)/p_{st}(A) = p(A,t;B,0)/p_{st}(A)$$
$$= p(A,0;B,-t)/p_{st}(A) = p(B,-t|A). \tag{26}$$

Similarly we can show that $p(A,t|B)=p(A,-t|B)$, thus the conditional probability x and y in Eq. (21) turned out to be even. Since $p(\sigma_1=1,t|A)=p(1,1,t|A)+p(1,-1,t|A)$, we see that $p(\sigma_1=1,t|A)$ is also even. Similarly $p(\sigma_1=-1,t|C)$ is even.

Now we are in a position to calculate the stationary distribution function explicitly. When t is small enough not to create correlation between $\sigma_1(t)$ and $\sigma_2(t)$, we can utilize statistical independence of $p(s_1^1,t|s_0^1,s_0^2)$ and $p(s_1^2,t|s_0^1,s_0^2)$ to have

$$p(s_1^1,s_1^2,t|s_0^1,s_0^2) = p(s_1^1,t|s_0^1,s_0^2)p(s_1^2,t|s_0^1,s_0^2). \tag{27}$$

To be more specific, $p(\sigma_1=1,t|A)$ is given as an even solution to Eq. (17) which satisfies the initial condition $p_1(1,t=0)=1$, $p_2(1,t=0)=1$, and $p(\sigma_1=1,t|A)=p(1,t)$. $p(\sigma_1=1,t|B)$ is similarly given as an even solution to Eq. (17) with $p_1(1,t=0)=1$, $p_2(1,t=0)=0$. Following a similar procedure as in Ref. [10], we have for $0<t<\tau$

$$p(\sigma_1=1,t|A) = [1+G(t)]/2,$$

$$p(\sigma_1=1,t|B) = [1+H(t)]/2, \tag{28}$$

where

$$G(t) = \frac{\{p\exp(-\lambda t) + m\exp[\lambda(t-\tau)]\}}{[p+m\exp(-\lambda\tau)]},$$

$$H(t) = \frac{\{p\exp(-\lambda t) - m\exp[\lambda(t-\tau)]\}}{[p-m\exp(-\lambda\tau)]}, \tag{29}$$

with

$$p = \sqrt{\gamma_1}+\sqrt{\gamma_2}, \quad m = \sqrt{\gamma_2}-\sqrt{\gamma_1}, \quad \lambda = 2\sqrt{\gamma_1\gamma_2}. \tag{30}$$

With these results we now have for small t

$$p(B,t|A) = [1+G(t)][1-G(t)]/4,$$

$$p(A,t|B) = [1+H(t)][1-H(t)]/4. \tag{31}$$

From Eq. (25), we have thus after lengthy calculations

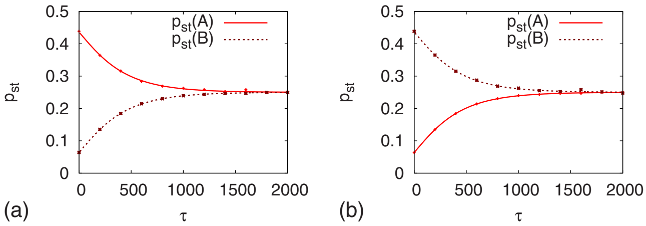

FIG. 1. (Color online) Stationary distribution functions $p_{st}(A)$ (solid) and $p_{st}(B)$ (dotted) from simulations (crosses) and theory (curves) as a function of the delay τ [Eq. (34)]. For the parameter values γ_1 and γ_2 we refer to Ref. [14] and [(a) and (b)] we use the same values hereafter.

$$p_{st}(A)/p_{st}(B) = [1 - H^2(t)]/[1 - G^2(t)] = F_A^2/F_B^2, \quad (32)$$

with

$$F_A = [p + m \exp(-\lambda\tau)],$$

$$F_B = [p - m \exp(-\lambda\tau)]. \quad (33)$$

It is amusing that the ratio [Eq. (32)] becomes independent of t, thus, from $p_{st}(A)+p_{st}(B)=1/2$, our method gives an exact stationary distribution, to be checked in Sec. III C.

$$p_{st}(A) = F_A^2/[2(F_A^2 + F_B^2)],$$

$$p_{st}(B) = F_B^2/[2(F_A^2 + F_B^2)]. \quad (34)$$

3. TCF

From the above it is readily seen that Eq. (18) is reduced to

$$\psi_s(t) = 4p_{st}(A)p(\sigma_1 = 1,t|A) + 4p_{st}(B)p(\sigma_1 = 1,t|B) - 1. \quad (35)$$

$$\psi_c(t) = 4p_{st}(A)p(\sigma_1 = 1,t|A) + 4p_{st}(B)p(\sigma_1 = -1,t|B) - 1. \quad (36)$$

We can extend these functions to the region $2\tau > t > \tau$ with Eq. (28) as inputs to Eq. (17). The stationary distribution function $p_{st}(A)$ and $p_{st}(B)$, which must be determined to calculate the TCF [Eqs. (35) and (36)] are given by Eq. (34).

C. Two-state system in the case $N=2$: Numerics

The stationary distribution function $p_{st}(A)$ and $p_{st}(B)$ obtained by Monte Carlo simulations (crosses) and from the theory [Eq. (34)] (curves) are shown in Fig. 1. When $\tau=0$ and $\epsilon>0$, there is a direct attractive force between $\sigma_1(t)$ and $\sigma_2(t)$ and two spins take the configuration $A=(1,1)$ more often than $B=(1,-1)$, thus $p_{st}(A)>p_{st}(B)$ [Fig. 1(a)]. As τ becomes large, this correlation becomes weak since the attraction is between $\sigma_1(t)$ and $\sigma_2(t-\tau)$ and thermal noise acting between $t-\tau$ and t tends to destroy the correlations. As we observe in Fig. 1 $p_{st}(A)\to 1/4$ and $p_{st}(B)\to 1/4$ as $\tau\to\infty$. When $\epsilon<0$ the interaction between the spin becomes repulsive and $p_{st}(A)<p_{st}(B)$ [Fig. 1(b)]. It is noted that our theory reproduces simulational results quite well.

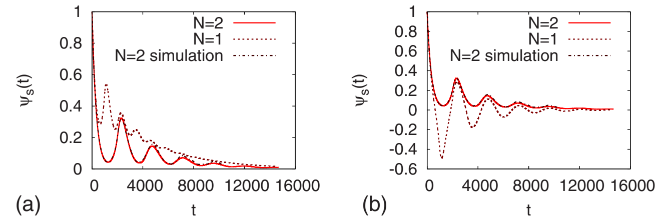

FIG. 2. (Color online) The TCF $\psi_s(t)$ is plotted for $N=1$ (dotted) and $N=2$ [solid and dashed (simulation)] as a function of t for $\tau=1000$ [14] and $D=0.005$ [(a) $\epsilon=+0.05$ and (b) $\epsilon=-0.05$] [Eq. (35)].

Next, we turn to the TCF $\psi_s(t)$ [Fig. 2] and $\psi_c(t)$ [Fig. 3]. It is noted that, when $\epsilon>0$, $\sigma_1(t)$ is positively correlated with $\sigma_2(t-\tau)$, which in turn is positively correlated with $\sigma_1(t-2\tau)$. This is the reason we observe positive correlations in $\psi_s(t)$ around $t\simeq 2\tau$ as a bump [Fig. 2(a)]. When ϵ is negative, two negative correlations are easily seen to produce positive correlations [Fig. 2(b)]. This may be the reason for the similarity between $\psi_s(t)$ for positive and negative ϵ.

Next we plot $\psi_c(t)$ for $\epsilon=0.05$ [Fig. 3(a)] and $\epsilon=-0.05$ [Fig. 3(b)]. We observe positive (negative) correlations in $\psi_c(t)$ as the convex (concave) at $t\simeq\tau$ for $\epsilon>0(<0)$, which is also observed for the case $N=1$. After the first bump in $\psi_c(t)$ similar but weaker bumps are seen about $\tau'\simeq 2\tau$ apart as in Fig. 2 for $\psi_s(t)$. Due to fluctuations τ' becomes larger than 2τ. Agreement between simulations and theory is seen to be excellent.

D. Two-state systems in the case $N=3$

We can calculate the stationary distribution function and the TCF in the case $N=3$, following the similar route as for the case $N=2$. From the symmetry of the system with respect to inversion $s_1,s_2,s_3\to -s_1,-s_2,-s_3$, we have only two stationary distribution functions. If we define the three-particle states K and L by

$$K:\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 1,$$

$$L:\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = -1, \quad (37)$$

we have

$$p_{st}(K) = F_A/[2(F_A + 3F_B)],$$

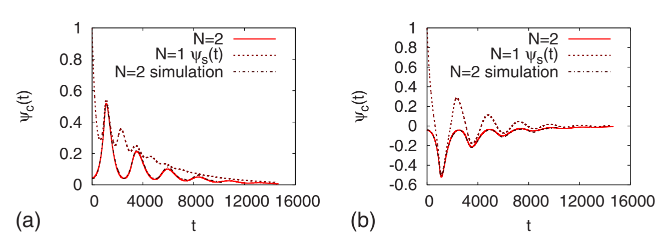

FIG. 3. (Color online) The TCF $\psi_c(t)$ is plotted for $N=1$ (dotted) and $N=2$ [solid and dashed(simulation)] as a function of t for $\tau=1000$ [14] and $D=0.005$ [(a) $\epsilon=+0.05$ and (b) $\epsilon=-0.05$] [Eq. (36)].

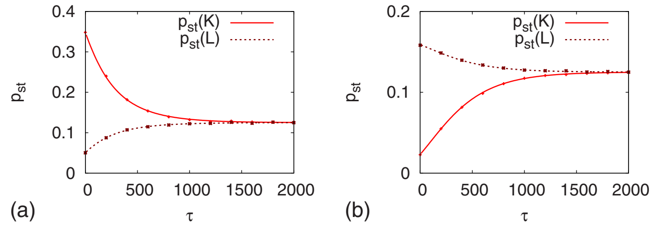

FIG. 4. (Color online) The stationary distribution functions $p_{st}(K)$ (solid) and $p_{st}(L)$ (dotted) as a function of τ [14] [(a) $\epsilon=+0.05$ and (b) $\epsilon=-0.05$] [Eq. (38)].

$$p_{st}(L) = F_B/[2(F_A + 3F_B)], \tag{38}$$

where F_A and F_B are defined by Eq. (33).

The TCF $\psi_{1,k}(t) \equiv \langle \sigma_1(t)\sigma_k(0)\rangle (k=1,2,3)$ is given after lengthy calculations, as in the case $N=2$, by

$$\psi_{1,k}(t) = \{p^{4-k}[m \exp(-\lambda\tau)]^{k-1} \exp(-\lambda t) + p^{k-1}[m \exp(-\lambda\tau)]^{4-k} \exp(\lambda t)\}/\{p^3 + [m \exp(-\lambda\tau)]^3\}, \tag{39}$$

where p, m, and λ are defined by Eq. (30).

As for the stationary distribution functions, plotted in Fig. 4, $p_{st}(K)$ is larger than $p_{st}(L)$ when $\epsilon>0$ due to the attractive interaction between the particles as in the case $N=2$ (see Fig. 1). When τ becomes large both $p_{st}(K)$ and $p_{st}(L)$ approaches 1/8 since there are 8 states for $N=3$.

The TCF are plotted in Fig. 5. For a positive ϵ [Fig. 5(a)], we notice two characteristic features. One is that the interval of the peaks is about 3τ (actually slightly longer than 3τ due to fluctuations). The other is that the first peak of $\psi_{1,i}(t)$ is located around $t\simeq(4-i)\tau$. These results are well understood in terms of delay attractive interaction between the particles $(i-1)$ and i. For a negative ϵ [Fig. 5(b)], we notice that the correlations observed between the bumps 3τ apart is negative. The first concave (around $t\simeq\tau$) and convex bumps (around $t\simeq 2\tau$) in $\psi_{1,2}(t)$ and $\psi_{1,3}(t)$, respectively, are also readily seen from the negative correlation between neighboring particle with time τ apart. As before, theory and simulations agree quite well.

IV. ANALYTIC APPROACH TO p_{st} AND TCF IN THE GENERAL N-PARTICLE CASE

In view of the lengthy calculations for $N=3$ necessary to have the exact results [Eqs. (38) and (39)], it became rather obvious that to pursue a similar approach for larger N was not useful. Thus our strategy is first to infer the general form for the stationary distribution and the TCF, which are next to be confirmed.

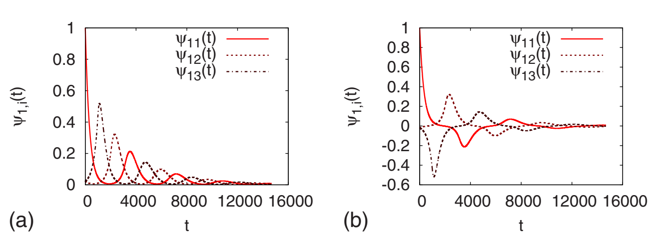

FIG. 5. (Color online) The TCF $\psi_{1,i}(t)[i=1(\text{solid}), 2(\text{dotted}), 3(\text{dashed})]$ as a function of t for $\tau=1000$ [(a) $\epsilon=+0.05$ and (b) $\epsilon=-0.05$ [14]] [Eq. (39)]. We omitted the figures of simulations because they are almost the same ones as the analytical ones.

A. Stationary distribution functions

From Eq. (10) and the solutions for the cases $N=1,2,3$, we notice that there are only four types of conditional probabilities for $0<t<\tau$, which are

$$p(\sigma_k = s_1^k, t | s_0^1, s_0^2, \ldots, s_0^N) = [1 \pm G(t)]/2,$$

$$\text{for } s_1^k = \pm s_0^k, s_0^k = s_0^{k-1},$$

$$p(\sigma_k = s_1^k, t | s_0^1, s_0^2, \ldots, s_0^N) = [1 \pm H(t)]/2,$$

$$\text{for } s_1^k = \pm s_0^k, s_0^k \neq s_0^{k-1}, \tag{40}$$

where $H(t)$ and $G(t)$ are defined by Eq. (29). For the time region $-\tau<t<0$, we have two cases. If the condition $s_0^{k-1} = s_0^{k+1}$ is satisfied, the conditional probabilities are even in t and t on the right-hand side of Eq. (40) needs to be replaced with $-t$. If this condition is not met, $[1\pm G(t)]/2$ on the right-hand side of Eq. (40) should be replaced by $[1\pm H(t)]/2$ and $[1\pm H(t)]/2$ should be replaced by $[1\pm G(t)]/2$. These solutions are easily confirmed to be valid by substituting them into Eq. (10).

We can calculate the transition probabilities for the general case by following the similar steps presented in Sec. III B 2 for $N=2$. That is, as was done in Eq. (27) we have for a small time region,

$$p(s_1^1, s_1^2, \ldots, s_1^N, t | s_0^1, s_0^2, \ldots, s_0^N) = \prod_{k=1}^{N} p(s_1^k, t | s_0^1, s_0^2, \ldots, s_0^N). \tag{41}$$

$p_{st}(s_0^1, s_0^2, \ldots, s_0^N)$ are inferred to be given by

$$p_{st}(s_0^1, s_0^2, \ldots, s_0^N) = Q_N(F_B^2/F_A^2)^k = Q_N\{[1-G(t)^2]/[1-H(t)^2]\}^k,$$

$$Q_N = (F_A/2)^N/\{p^N + [m \exp(-\lambda\tau)]^N\}, \tag{42}$$

where F_A and F_B are defined by Eq. (32). Equation (42) shows that the distribution function depends only on the number k with $2k$ denoting the number of interfaces present in the spin configuration $s_0^1, s_0^2, \ldots, s_0^N$. When s_0^i and s_0^{i+1} are different, we consider that there is an interface between the spin i and $i+1$. Since we assume that N spins are put on a circle, we consider that there is an interface if s_0^1 and s_0^N are different. Then the number of interfaces produced by the spin configuration is always even, which we denote by $2k$.

Q_N in Eq. (42) is a normalization constant. At this point we introduce a technique useful for confirmation of our inferred solutions for p_{st} [Eq. (42)] and the TCF [Eq. (53)] below. Let us introduce the binary numbers $b_i=\pm 1(i=1,2,\ldots,N)$ which are defined by

$$b_i = (s_0^i + 1)/2. \tag{43}$$

Then the number of interfaces $2k$ is expressed with $s_{N+1} = s_1$ as

$$2k = \sum_{i=1}^{N} [b_i(1 - b_{i+1}) + (1 - b_i)b_{i+1}]. \tag{44}$$

From Eqs. (42) and (44), we now see that Q_N^{-1} is the partition function Z for the N-body system

$$Z = \sum_{b_i=0,1} (F_B/F_A)^{\Sigma_{i=1,N}[b_i(1-b_{i+1})+(1-b_i)b_{i+1}]}. \tag{45}$$

With a traditional method such as the 2×2 transfer matrix one, it is easily confirmed that p_{st} in Eq. (42) actually satisfies the normalization condition.

With these preparations, we now explain how one can confirm the validity of the stationary distribution [Eq. (42)]. From Eq. (40) one can express the transition probability [Eq. (41)] in the form

$$p(s_1^1, s_1^2, \ldots, s_1^N, t | s_0^1, s_0^2, \ldots, s_0^N) = (1/2)^N [1 + G(t)]^l [1 - G(t)]^m \times [1 + H(t)]^n [1 - H(t)]^p, \tag{46}$$

where the integers l, m, n, p are given in term of $b_i (i = 1, 2, \ldots, N)$ [Eq. (43)] and $a_i (i = 1, 2, \ldots, N)$ defined by $a_i = (s_1^i + 1)/2$ as follows:

$$l = \sum_{i=1}^{N} [a_i b_i b_{i+1} + (1 - a_i)(1 - b_i)(1 - b_{i+1})],$$

$$m = \sum_{i=1}^{N} [(1 - a_i) b_i b_{i+1} + a_i (1 - b_i)(1 - b_{i+1})],$$

$$n = \sum_{i=1}^{N} [a_i b_i (1 - b_{i+1}) + (1 - a_i)(1 - b_i) b_{i+1}],$$

$$p = \sum_{i=1}^{N} [(1 - a_i) b_i (1 - b_{i+1}) + a_i (1 - b_i) b_{i+1}]. \tag{47}$$

The eigenvalue problem [Eq. (24)] which is generalized to the N dimensional case is written down as

$$\sum_{b_N=0}^{1} \cdots \sum_{b_1=0}^{1} (1/2)^N [1 + G(t)]^l [1 - G(t)]^m \times [1 + H(t)]^n [1 - H(t)]^p \times Q_N \{[1 - G(t)^2]/[1 - H(t)^2]\}^k = Q_N \{[1 - G(t)^2]/[1 - H(t)^2]\}^q, \tag{48}$$

where k is given by Eq. (44). Now what should be checked is whether the right-hand side of Eq. (48) corresponds to $p_{st}(s_1^1, s_1^2, \ldots, s_1^N)$, that is, whether the number q on the right-hand side of Eq. (48) is given by

$$q = (1/2) \sum_{i=1}^{N} [a_i(1 - a_{i+1}) + (1 - a_i) a_{i+1}]. \tag{49}$$

After lengthy calculations it is confirmed that Eq. (49) is really valid.

B. TCFs

Next let us consider the TCF in the range $0 < t < \tau$, which is defined by

$$\psi_{1k}^N(t) = \langle \sigma_1(t) \sigma_k(0) \rangle = \sum_{\sigma_1(t)=-1}^{+1} \sum_{\sigma_k(0)=-1}^{+1} \sigma_1 \sigma_k p_{st}(\sigma_1(t); \sigma_k(0)), \tag{50}$$

where

$$p_{st}(\sigma_1(t); \sigma_k(0)) \equiv \sum_{s_0^i (i \neq k)} p_{st}(s_0^1, \ldots, s_0^k = \sigma_k(0), \ldots, s_0^N) \times p(\sigma_1, t | s_0^1, \ldots, s_0^k = \sigma_k(0), \ldots, s_0^N) \tag{51}$$

By using Eq. (40), it is seen that $\psi_{1k}^N(t)$ is expressed as

$$\begin{aligned}\psi_{1k}^N(t) = \sum_{s_0^i} \{&G(t)[p_{st}(s_0^1 = 1, \ldots, s_0^k = 1, \ldots, s_0^N = 1) \\ &+ p_{st}(s_0^1 = 0, \ldots, s_0^k = 0, \ldots, s_0^N = 0) - p_{st}(s_0^1 = 1, \ldots, s_0^k = 0, \ldots, s_0^N = 1) - p_{st}(s_0^1 = 0, \ldots, s_0^k = 1, \ldots, s_0^N = 0)] \\ &+ H(t)[p_{st}(s_0^1 = 0, \ldots, s_0^k = 1, \ldots, s_0^N = 1) \\ &+ p_{st}(s_0^1 = 1, \ldots, s_0^k = 0, \ldots, s_0^N = 0) - p_{st}(s_0^1 = 0, \ldots, s_0^k = 0, \ldots, s_0^N = 1) - p_{st}(s_0^1 = 1, \ldots, s_0^k = 1, \ldots, s_0^N = 0)]\}.\end{aligned} \tag{52}$$

Here, we notice that the sum over $s_0^i = \pm 1$ in Eq. (52) is performed for $i = 2, \ldots, k-1, k+1, \ldots, N-1$. This part of the calculation is similar to the calculation of Z in the previous subsection and we arrive at the final results for the TCF of the form

$$\psi_{1k}^N(t) = \{p^{N+1-k}[m \exp(-\lambda\tau)]^{k-1} \exp(-\lambda t) + p^{k-1}[m \exp(-\lambda\tau)]^{N+1-k} \exp(\lambda t)\} / \{p^N + [m \exp(-\lambda\tau)]^N\} (k = 1, \ldots, N). \tag{53}$$

It is readily seen that the general results [Eq. (42)] for p_{st} and [Eq. (53)] for the TCF, reduce to the corresponding results for the cases $N=2$ and $N=3$, presented in Secs. III B and III D, respectively.

In passing we give one remark as to the summation $\Sigma_{s_0^i}$ in Eq. (52). When k is 1, this sum is over s_0^i with $i = 2, 3, \ldots, N-1$. Similarly when $k=N$, we sum over s_0^i with $i = 2, 3, \ldots, N-1$.

V. SOME REMARKS

In this paper we considered stochastic dynamics in many-body systems, where two-state particles are unidirectionally coupled with delay. For a system consisting of few particles, we found analytic expressions for the TCF and the p_{st}, which agree well with numerical experiments. Particle correlations, as observed from the p_{st} and various TCF's, could be interpreted based on correlations between neighboring particles. In this sense one can say that the correlations due to feedback for the case $N=1$ are the building blocks for correlations in many particle systems.

From p_{st} for the general N-body problem [Eqs. (42) and (45)] we have an interesting map from a system with unidirectional coupling to a system with nearest-neighbor coupling. That is, let us express Z of Eq. (45) as

$$Z = \sum_{b_i=0,1} \exp[2k \ln(F_B/F_A)] = \sum_{b_i=0,1} \exp[-E|\ln(F_B/F_A)|]. \tag{54}$$

Here E, the energylike quantity of the system is

$$E = 2k(\epsilon > 0), \quad E = -2k(\epsilon < 0), \tag{55}$$

where use is made of Eqs. (30) and (32) to decide the sign of $\ln(F_B/F_A)$. Thus it is seen that the energy of an interface is positive for $\epsilon > 0$ and negative for $\epsilon < 0$. Since there are $2 \times N!/[(2k)!(N-2k)!]$ ways of putting $2k$ interface on the N spin system, with the factor of 2 denoting the two possibilities of $s_1 = \pm 1$, one can estimate the most probable number of the interface $2k_{m.p}$ by maximizing, with $\omega \equiv F_B/F_A$,

$$G = \omega^{2k} N!/[(2k)!(N-2k)!]. \tag{56}$$

It is seen that

$$2k_{m.p}/N = \omega/[1+\omega], \tag{57}$$

which is plotted in Fig. 6. For $\omega=1$, corresponding to $\tau=0$, i.e., $\gamma_1=\gamma_2$ and $m=0$ [Eq. (30)], or $\tau \to \infty$, we have $2k_{m.p}/N=1/2$ since interface energy is zero and only the entropy matters. As interface energy becomes large (ω is decreasing from 1), interface number decreases. These effects are already seen in Figs. 1 and 4.

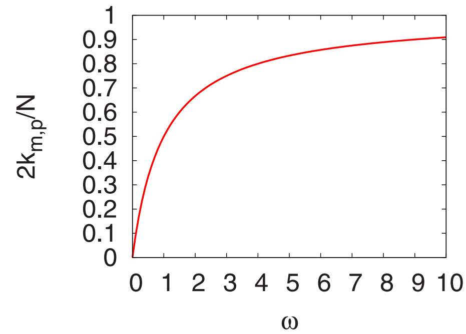

FIG. 6. (Color online) The most probable number of the interface as a function of $\omega=F_B/F_A$ [Eq. (57)].

It is noted here that for the unidirectional coupling with delay, one cannot see any transition phenomena in the large N limit. On the other hand, for the global coupling with delay, some kind of transitions, such as ordering phase transitions and appearance of oscillatory behaviors due to a Hopf bifurcation, are reported in the limit $N \to \infty$ [15]. It may be of some interest to see how our approach could be applied to study other stochastic models with more realistic feedback interaction.

Finally, we note that the circular boundary condition employed in our model made the behavior of the system rather simple. Without these conditions, the particle symmetry, mentioned just below Eq. (21), does not hold and consequently the number of independent p_{st} and TCF will be of the order of N^2. At the moment we could not find analytic solutions for this case.

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